Grade of Service of Direct Traffic Mixed with Store-and-Forward Traffic*

By JOSEPH OTTERMAN†

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The dual use of trunks for both direct and store-and-forward (S/F) traffic makes high trunk efficiency possible. The resultant trunk savings are important in communication systems in which long-haul trunks contribute heavily to the cost of the system. This paper reports work on the computation of trunking tables that could be used to engineer trunk requirements for prescribed loads and distributions of direct and S/F traffic.

A method of computation and some specific results in terms of grade of service of direct traffic and traffic capacity for S/F traffic are presented. The numerical results are for two to forty-eight trunks. The results apply to the case of exponentially-distributed holding times of both the direct and the S/F traffic.

I. INTRODUCTION

Direct traffic is a user-to-user service (generally voice) requiring a connection to be established promptly on demand. Store-and-forward (S/F) traffic, on the other hand, is stored at or near the originator's location and is later sent to its destination, either directly or through further intermediate storage. If S/F traffic is sent only when the direct traffic load is light, large amounts of S/F traffic can be accommodated with very slight degradation of direct traffic service. This can be especially important in a long-haul communication system where the trunk group cross sections (number of trunks in a group) are small. Such trunk groups are notoriously inefficient if used only for direct traffic, but they have a substantial ability to handle additional S/F traffic.

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† I.T.T Federal Laboratories, Nutley, N. J., (work performed for Bell Telephone Laboratories). Present affiliation, General Electric Company.

The basic operating method analyzed in this paper is as follows. The occupancy of the trunks in a trunk group is monitored, and whenever the occupancy drops below a certain level, the S/F traffic is allowed access to the idle trunks. The sending of an S/F message is *not* interrupted in order to service arrivals of direct traffic, but when transmission of an S/F message is begun, a specified number of trunks is always held in reserve for direct traffic. Under the foregoing rules, the present analysis establishes (*i*) the grade of service of direct traffic and (*ii*) the amount of S/F traffic that can be accommodated. Direct traffic congestion discipline is assumed to be governed by the lost-calls-cleared assumption,* which means that direct calls which encounter an all-trunks busy condition do not reappear in the busy hour.

The statistical nature of traffic is first summarized. The problem of dual use of trunks is formulated as a probabilistic net representing a Markov chain. It is assumed that a queue of S/F traffic exists at all times. Using cut-sets in the state graph, this probabilistic net is analyzed to derive expressions for the steady-state probabilities of trunk occupancy. From the expressions of trunk occupancy for a given load of direct traffic, the grade of service of the direct traffic and the amount of the S/F traffic that can be accommodated are determined. A glossary of symbols is given in the appendix.

II. MATHEMATICAL ANALYSIS

2.1 *Statistical Properties of Traffic*

The direct calls are assumed to be generated individually and collectively at random. The expected number of arrivals per hour is denoted as n. The expected number of arrivals during a fraction of an hour $d\tau$ is simply given by $n\ d\tau$. When $d\tau$ is short enough so that multiple events are improbable, the probability of an arrival during $d\tau$ is equal to the expected number of arrivals: i.e., equal to $n\ d\tau$.

In the following discussion it will be more convenient to measure time in fractions, dt of the average holding time of the direct traffic, which is denoted T. Then the probability of an arrival during a short time, dt, is therefore:

$$\text{Probability of an arrival during } dt = nT\ dt = a\ dt. \tag{1}$$

* The assumption of lost-calls-cleared is especially appropriate when an alternate route is provided for calls that encounter the all-trunks-busy condition. When an alternate route is provided, the probability of loss should be interpreted as the portion of calls that overflow, seeking the alternate route.

The product nT, which is denoted a, denotes the offered direct traffic measured in erlangs. It represents the expected number of calls in progress on a fully served basis. Equation (1) indicates still another significance of a: it is the expected number of arrivals during one holding time.

The holding times of both the direct and the S/F traffic are assumed to be exponentially distributed (with average T for the direct and t_0 for S/F traffic). Under the assumption of exponential holding times, the probability of termination of a call with average holding time T during a differential interval $d\tau$ is given by $d\tau/T$ plus terms negligible in the limit $d\tau \to 0$. In fractions dt of the holding time T, this probability is simply dt. When x calls are in progress, the probability of the termination on one of these calls during a differential interval dt is given by:

$$\text{Probability of a termination out of } x \text{ calls} = x\,dt. \tag{2}$$

Similarly, the probability of a termination out of z S/F messages in transmission is $rz\,dt$, where r is the ratio T/t_0.

The probability of a termination during dt depends on the present status of the trunk group: i.e., the number of calls in progress. Neither the probability of an arrival (1), nor the probability of a termination (2), depends upon the past history of the calls. This demonstrates the fact that under the above conditions the trunk occupancy is a Markov process.

The assumption of lost-calls-cleared is used; that is, a direct call that encounters a condition of all-trunks-busy is cleared from that trunk group and does not reappear in the busy hour. In the absence of S/F traffic, the trunk occupancy is a relatively simple Markov process, which is shown in the flow diagram in Fig. 1. Statistical equilibrium considerations lead to the following formula for the probability G_x of exactly x trunks being busy:

$$G_x = \frac{\dfrac{a^x}{x!}}{\displaystyle\sum_{y=0}^{c} \frac{a^y}{y!}}. \tag{3}$$

The probability of loss (probability of all-trunks-busy) is the well known Erlang B formula:[2]

$$B(c,a) = \frac{\dfrac{a^c}{c!}}{\displaystyle\sum_{y=0}^{c} \frac{a^y}{y!}}. \tag{4}$$

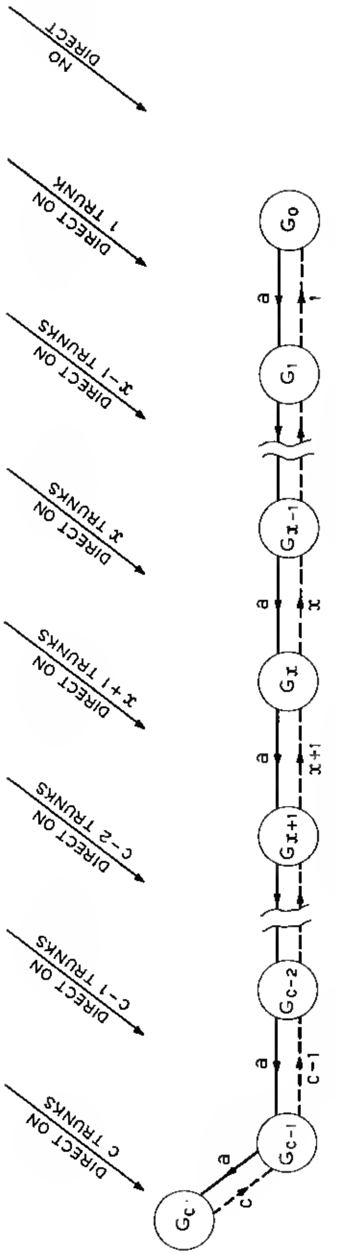

Fig. 1 — Markov chain for trunk occupancy under lost-calls-cleared assumption, no S/F traffic.

2.2 *Trunk Occupancy by Direct and S/F Traffic on a Lost-Calls-Cleared Basis*

2.2.1 *One Trunk in Reserve, $s = 1$*

The queue of S/F traffic is given access to the trunk group only when the occupancy is such that one trunk will remain idle after transmission of an S/F message is initiated. An S/F queue is assumed to exist at all times, which means that whenever two trunks in the trunk group become idle, one of them is taken for S/F transmission. When the reserve trunk becomes busy with direct traffic, S/F transmission is stopped on the trunk on which the next (in time sequence) termination of an S/F message occurs. This last-mentioned trunk then becomes a reserve trunk.

The occupancy of trunks can be regarded as a Markov process, which is shown in the flow diagram of Fig. 2. The circles in the bottom row represent states in which one trunk is available to accommodate possible arrivals of direct traffic. Conversely, the circles in the top row represent all-trunks-busy conditions. Trunk occupancy by S/F traffic is represented by the index z, which, in Fig. 2, increases from left to right. S_z represents the steady-state probability of z trunks busy with S/F traffic (in the bottom row): i.e., $(c - z - 1)$ trunks busy with direct traffic and one trunk in reserve. R_z represents the steady-state probability of z trunks busy with S/F traffic (in the top row): i.e., $(c - z)$ trunks busy with direct traffic and no trunks idle.

In the steady state, the transition probabilities out of S_z during dt are: $a\, dt\, S_z$ through a new arrival (transition into R_z) and $(c - z - 1)$ $dt\, S_z$ through a termination of one of $(c - z - 1)$ direct calls in progress (into S_{z+1} since, as soon as two trunks are recognized as idle, one of them is seized for S/F transmission).

Transition probabilities into S_z are: $(c - z)dtR_z$ through a termination of one of $(c - z)$ direct calls in progress in state R_z (in the time interval after an arrival of a direct call and before the next termination, in time, of an S/F message); $(z + 1)r\, dtR_{z+1}$ through a termination of one of the $(z + 1)$ S/F messages in progress in state R_{z+1}; and $(c - z)$ $dt\, S_{z-1}$ through a termination of one of the $(c - z)$ direct calls in progress in state S_{z-1}. Equating the transition probabilities out of S_z to transition probabilities into S_z results in the following equation:

$$(a + c - z - 1)S_z = (c - z)S_{z-1} + (z + 1)r\, R_{z+1} + (c - z)R_z . \quad (5)$$

The transition probabilities out of R_z during dt are: $(c - z)$ $dt\, R_z$

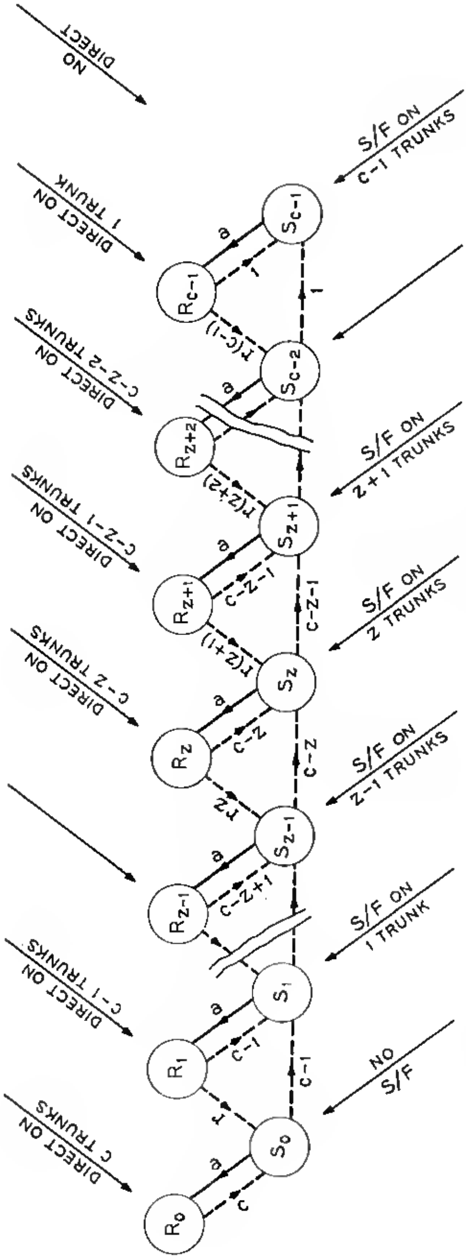

Fig. 2 — Markov chain for trunk-occupancy under lost-calls-cleared assumption, dual use of trunks, one trunk in reserve.

through a termination of one of $(c - z)$ direct calls in progress (into S_z) and $zr\, dt\, R_z$ through a termination of one of z S/F messages in progress (into S_{z-1}). The transition probability into R_z is $adt\, S_z$ through an arrival of a direct call.

It should be pointed out here that if a new, direct arrival occurs when the chain is in state R_z, one call is lost (or takes another route), but transition to another state does not occur under the assumption of lost-calls-cleared.

Equating the transition probabilities into R_z to transition probabilities out of R_z results in the following equation:

$$aS_z = (c - z + zr)R_z = [c + z(r - 1)]R_z . \tag{6}$$

Subtracting (6) from (5) results in

$$(z + 1)rR_{z+1} = (c - z - 1)S_z - (c - z)S_{z-1} + zr\, R_z . \tag{7}$$

This relation is satisfied if the following holds:

$$(z + 1)rR_{z+1} = (c - z - 1)S_z \tag{8}$$

and therefore,

$$zrR_z = (c - z)S_{z-1} . \tag{9}$$

Actually, the recurrence relation of (8) and (9) can be observed directly by equating the transition probabilities through the appropriate cut-set in Fig. 2.

We will express first R_z and S_z, $0 \leqq z \leqq (c - 1)$, in terms of R_0, the probability that all trunks are busy with direct traffic. Combining (6) and (8) shows that

$$ar(z + 1)R_{z+1} = (c - z - 1)[c + z(r - 1)]R_z \tag{10}$$

and therefore,

$$arzR_z = (c - z)[c + (z - 1)(r - 1)]R_{z-1} . \tag{11}$$

Iteration of (11) results in the following expression for R_z:

$$R_z = \frac{[c + 0(r - 1)][c + 1(r - 1)][c + 2(r - 1)] \cdots [c + (z - 1)(r - 1)]}{z!\, r^z} \cdot \frac{(c - 1)(c - 2) \cdots (c - z)}{a^z} R_0 . \tag{12}$$

From (12) and (6),

$$S_z = \frac{[c + 0(r-1)][c + 1(r-1)][c + 2(r-1)] \cdots [c + z(r-1)]}{z!\, r^z} \cdot \frac{(c-1)(c-2)\cdots(c-z)}{a^{z+1}} R_0 . \tag{13}$$

Now, R_0 can be determined for a given load a by noting that the sum of probabilities R_z and S_z adds up to unity:

$$\sum_{z=0}^{c-1} R_z + \sum_{z=0}^{c-1} S_z = 1. \tag{14}$$

Introducing R_z and S_z from (12) and (13), respectively,

$$\begin{aligned} \frac{1}{R_0} &= \sum_{z=0}^{c-1} \frac{[c + 0(r-1)][c + 1(r-1)][c + 2(r-1)] \cdots [c + (z-1)(r-1)]}{z!\, r^z} \cdot \frac{(c-1)(c-2)\cdots(c-z)}{a^z} \\ &\quad + \sum_{z=0}^{c-1} \frac{[c + 0(r-1)][c + 1(r-1)][c + 2(r-1)] \cdots [c + z(r-1)]}{z!\, r^z} \cdot \frac{(c-1)(c-2)\cdots(c-z)}{a^{z+1}} \end{aligned} \tag{15}$$

$$= \sum_{z=0}^{c-1} \{a + c + z(r-1)\} \cdot \frac{[c + 0(r-1)][c + 1(r-1)] \cdots [c + (z-1)(r-1)]}{z!\, r^z} \cdot \frac{(c-1)(c-2)\cdots(c-z)}{a^{z+1}} .$$

The probability of loss corresponds to the probability that all trunks are busy: i.e., the sum of the probabilities R_z. This can be referred to as the dual-use-of-trunks formula, $D(c,a,r,1)$ for c trunks, a erlangs of offered direct load, ratio r of holding times, and one trunk in reserve.

$$D(c,a,r,1) = \sum_{z=0}^{c-1} R_z$$

$$= \frac{\displaystyle\sum_{z=0}^{c-1} \frac{[c + 0(r-1)][c + 1(r-1)] \cdots [c + (z-1)(r-1)]}{z!\, r^z} \cdot \frac{(c-1)(c-2)\cdots(c-z)}{a^z}}{\displaystyle\sum_{z=0}^{c-1} [a + c + z(r-1)] \cdot \frac{[c + 0(r-1)][c + 1(r-1)] \cdots [c + (z-1)(r-1)]}{z!\, r^z} \cdot \frac{(c-1)(c-2)\cdots(c-z)}{a^{z+1}}}. \tag{16}$$

The amount of S/F traffic in erlangs that can be accommodated is denoted b. It is given by

$$b = \sum_{z=0}^{c-1} zS_z + \sum_{z=0}^{c-1} zR_z. \tag{17}$$

Equations (16) and (17) constitute the results sought.

The load b can be determined by a simpler formula when the grade of service $D(c,a,r,1)$ has been calculated. This formula can be arrived at by the following reasoning. The load in terms of trunks occupied is $c - 1$ at all times, plus one additional trunk with probability $D(c,a,r,1)$. The erlang load in terms of traffic carried is b for the S/F traffic and $a[1 - D(c,a,r,1)]$ for the direct traffic. [The portion $aD(c,a,r,1)$ is cleared under the assumption of lost-calls-cleared.] Equating the two results yields the following equation,

$$b + a[1 - D(c,a,r,1)] = c - 1 + D(c,a,r,1) \tag{18}$$

and therefore

$$\begin{aligned} b &= c - 1 - a[1 - D(c,a,r,1)] + D(c,a,r,1) \\ &= c - (a+1)[1 - D(c,a,r,1)]. \end{aligned} \tag{19}$$

It should be pointed out that for $r \to \infty$ the probabilities $R_z \to 0$ as $1/r$ (except for R_0). In this limiting case the probability of loss tends to $B(c,a)$ with the difference—i.e., the impairment due to the S/F traf-

fic—being of the order of $1/r$. This can be written

$$\lim_{r\to\infty} [D(c,a,r,1) - B(c,a)] = 0\left(\frac{1}{r}\right). \tag{20}$$

The probabilities R_0, S_0, S_1, $\cdots$ S_{c-1} tend to become, respectively, G_c, G_{c-1}, G_{c-2} $\cdots$ G_0.

2.2.2 *Two Trunks in Reserve, $s = 2$*

In this case, the queue of S/F traffic is denied access to the trunk group unless two trunks will remain idle after transmission of an S/F message is initiated. A queue of S/F traffic is again assumed to exist at all times, which means that as soon as three trunks in the trunk group become idle, one of them is taken for transmission of an S/F message. When one (or two) of the reserve trunks becomes busy with the direct traffic, S/F transmission is stopped on the trunk on which the next (in time sequence) termination of an S/F message occurs. The flow diagram of the process is shown in Fig. 3. In states S, two trunks are idle; in states W, one trunk is idle; in states R, all trunks are busy. Thus, in state S_z, z trunks are busy with the S/F traffic and $(c - z - 2)$ with the direct traffic; in state W_z, z trunks are busy with the S/F traffic and $(c - z - 1)$ with the direct traffic; in state R_z, z trunks are busy with the S/F traffic and $(c - z)$ with the direct traffic. The transition probability coefficients are as given in the Fig. 3.

Three basic equations are presented now, which state that the net transition flow through three cut-sets is zero under the steady-state conditions. The three cut-sets are indicated in Fig. 3 by lines of dots.

$$aS_z + (z + 1)rR_{z+1} = (c - z - 1)W_z + (c - z - 1)S_{z-1} \tag{21}$$

$$(c - z - 1)S_{z-1} = zrR_z + zrW_z \tag{22}$$

$$(c - z + zr)R_z = [c + z(r - 1)]R_z = aW_z. \tag{23}$$

On the basis of (23), (22) can be rewritten as follows:

$$\begin{aligned}(c - z - 1)S_{z-1} &= zr\left[1 + \frac{c + z(r - 1)}{a}\right]R_z \\ &= zr\left[1 + \frac{a}{c + z(r - 1)}\right]W_z\end{aligned} \tag{24}$$

and similarly,

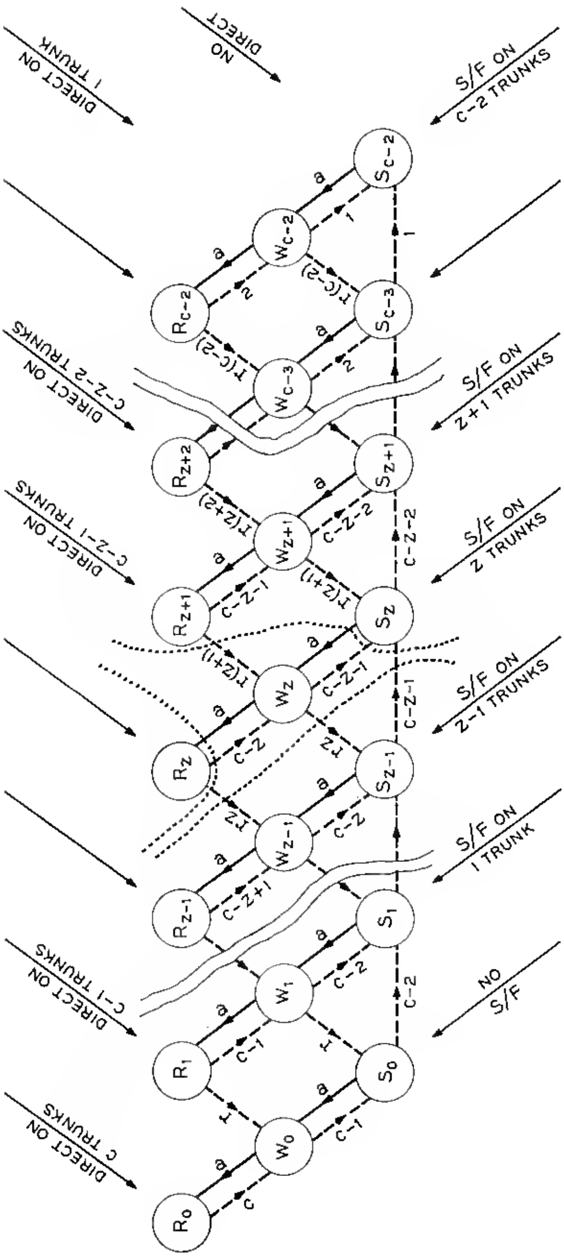

Fig. 3 — Markov chain for trunk occupancy under lost-calls-cleared assumption, dual use of trunks, two trunks in reserve.

$$
\begin{aligned}
(c - z - 2)S_z &= (z + 1)r\left[1 + \frac{c + (z+1)(r-1)}{a}\right]R_{z+1} \\
&= (z + 1)r\left[1 + \frac{a}{c + (z+1)(r-1)}\right]W_{z+1}.
\end{aligned} \tag{25}
$$

Introducing the relation from (25) into the left side of (21) and the relation from (24) into the right side of (21), we obtain

$$
\begin{aligned}
a\left[1 + \frac{c - z - 2}{a + c + (z+1)(r-1)}\right]S_z \\
&= \left[(c - z - 1) + zr\left(1 + \frac{a}{c + z(r-1)}\right)\right]W_z \qquad (26) \\
&= \left[c + z(r-1) - 1 + \frac{zra}{c + z(r-1)}\right]W_z.
\end{aligned}
$$

We assume R_0 is known and proceed to derive the other probabilities in the following manner: W_z is determined from R_z through the use of (23), which is rewritten in a convenient form as (27). S_z is determined from W_z through the use of (26), which is rewritten in a convenient form as (28). R_{z+1} is determined from S_z through the use of (25), which is rewritten in a convenient form as (29):

$$
W_z = \frac{[c + z(r-1)]}{a} R_z \tag{27}
$$

$$
S_z = \frac{\left[c + z(r-1) - 1 + \frac{zra}{c + z(r-1)}\right]}{a\left[1 + \frac{c - z - 2}{a + c + (z+1)(r-1)}\right]} W_z \tag{28}
$$

$$
R_{z+1} = \frac{(c - z - 2)a}{(z+1)r[a + c + (z+1)(r-1)]} S_z. \tag{29}
$$

We obtain, using (27), for $z = 0$,

$$
W_0 = \frac{[c + 0(r-1)]}{a} R_0. \tag{30}
$$

Using (28), for $z = 0$,

$$S_0 = \frac{\left[c + 0(r-1) - 1 + \dfrac{0ra}{c+0(r-1)}\right]}{a\left[1 + \dfrac{c-0-2}{a+c+1(r-1)}\right]} W_0$$

$$= \frac{\left[c + 0(r-1) - 1 + \dfrac{0ra}{c+0(r-1)}\right]}{\left[1 + \dfrac{c-0-2}{a+c+1(r-1)}\right]} \frac{[c+0(r-1)]}{a^2} R_0. \tag{31}$$

Using (29), for $z = 0$,

$$R_1 = \frac{(c-0-2)a}{1r[a+c+1(r-1)]} S_0$$

$$= \frac{\left[c + 0(r-1) - 1 + \dfrac{0ra}{c+0(r-1)}\right]}{\left[1 + \dfrac{c-0-2}{a+c+1(r-1)}\right]} \frac{[c+0(r-1)]}{a} \tag{32}$$

$$\cdot \frac{c-0-2}{1r[a+c+1(r-1)]} R_0.$$

Using (27), for $z = 1$,

$$W_1 = \frac{[c+1(r-1)]}{a} R_1. \tag{33}$$

Using (28), for $z = 1$,

$$S_1 = \frac{c + 1(r-1) - 1 + \dfrac{1ra}{c+1(r-1)}}{a\left[1 + \dfrac{c-1-2}{a+c+2(r-1)}\right]} W_1. \tag{34}$$

Finally

$$R_z = \prod_{k=0}^{z-1} \frac{c+k(r-1)}{a} \prod_{k=0}^{z-1} \frac{\left[c + k(r-1) - 1 + \dfrac{kra}{c+k(r-1)}\right]}{a\left[1 + \dfrac{c-k-2}{a+c+(k+1)(r-1)}\right]} \tag{35}$$

$$\cdot \prod_{k=0}^{z-1} \frac{(c-k-2)a}{(k+1)r[c+(k+1)(r-1)]} R_0$$

$$W_z = \prod_{k=0}^{z} \frac{c + k(r-1)}{a} \prod_{k=0}^{z-1} \frac{\left[c + k(r-1) - 1 + \dfrac{kra}{c + k(r-1)}\right]}{a\left[1 + \dfrac{c-k-2}{a+c+(k+1)(r-1)}\right]} \cdot \prod_{k=0}^{z-1} \frac{(c-k-2)a}{(k+1)r[c+(k+1)(r-1)]} R_0 \tag{36}$$

$$S_z = \prod_{k=0}^{z} \frac{c + k(r-1)}{a} \cdot \prod_{k=0}^{z} \frac{\left[c + k(r-1) - 1 + \dfrac{kra}{c + k(r-1)}\right]}{a\left[1 + \dfrac{c-k-2}{a+c+(k+1)(r-1)}\right]} \cdot \prod_{k=0}^{z-1} \frac{(c-k-2)a}{(k+1)r[c+(k+1)(r-1)]} R_0 . \tag{37}$$

Now R_0 can be determined, since

$$\sum_{z=0}^{c-2} R_z + \sum_{z=0}^{c-2} W_z + \sum_{z=0}^{c-2} S_z = 1. \tag{38}$$

The probability of loss $D(c,a,r,2)$ is equal to the probability that all trunks are busy

$$D(c,a,r,2) = \sum_{z=0}^{c-2} R_z . \tag{39}$$

When values for R_z obtainable from (35), (36), (37), and (38) are used in (39), it can be regarded as the dual-use-of-trunks formula $D(c,a,r,2)$ for c trunks, a erlangs of offered direct load, ratio of holding times r, and two trunks in reserve.

The amount b of the S/F traffic in erlangs that can be accommodated is given by

$$b = \sum_{z=0}^{c-2} zR_z + \sum_{z=0}^{c-2} zW_z + \sum_{z=0}^{c-2} zS_z . \tag{40}$$

It should be pointed out that for $r \to \infty$, the probabilities $R_z \to 0$ as $1/r^2$ (except for R_0) and the probabilities $W_z \to 0$ as $1/r$ (except for W_0). In this limiting case, the probability of loss tends to $B(c,a)$ with the difference — i.e., the impairment due to S/F traffic — being of the order $1/r^2$:

$$\lim_{r\to\infty} [D(c,a,r,2) - B(c,a)] = 0\left(\frac{1}{r^2}\right). \tag{41}$$

The probabilities R_0, W_0, S_0, S_1, $\cdots$ S_{c-2} tend to become, respectively,

G_c, G_{c-1}, G_{c-2}, G_{c-3} $\cdots$ G_0. The load b becomes

$$b = c - 2 - a[1 - D(c,a,r,2)] + 2G_c + G_{c-1}. \qquad (42)$$

This can be seen by the following reasoning. The load in terms of trunks occupied is $(c - 2)$ trunks at all times, one additional trunk with the probability G_{c-1} and two additional trunks with the probability G_c. The erlang load in terms of traffic carried is b for the S/F and $a[1 - D(c,a,r,2)]$ for the direct traffic. Equating the two results yields (42).

III. RESULTS AND CONCLUSIONS

The recurrence relations for trunk occupancy probabilities as derived above were programmed on an IBM 704 computer by Miss B. Berman. Two types of programs were written:

i. The grade of service of direct traffic and the amount of S/F traffic accommodated are calculated for a given amount a of direct traffic, a given number of trunks in a trunk group, and either one or two trunks in reserve.

ii. The amount a of direct traffic and the amount of S/F traffic are calculated for a given grade of service assigned to direct traffic, a given number of trunks in a trunk group, and either one or two trunks in reserve. This computer program is similar to (*i*) except that it involves successive approximations in the amount of direct traffic to obtain the desired grade of service.

These computer programs were used for trunk groups ranging in size from 2 to 48 trunks. Special attention was given to one ratio of the average holding time of direct traffic to the average holding time of S/F traffic, $r = T/t_0 = 300/3.6 = 83.3$. This ratio was selected to correspond to the average holding times which were originally expected in the UNICOM system, for which the study was initially done. For the direct traffic, $T = 5$ minutes $= 300$ seconds, and for the S/F traffic, $t_0 = 3.6$ seconds. This last figure was obtained by assuming 150 words for the average message length, 42-bits-per-word digital coding, and 2400-bits-per-second transmission speed. Approximately one second is allowed in the total for the average time required to establish the connection. Thus, on the average, for one fully loaded trunk, 1000 S/F messages per hour is the equivalent of only 12 direct calls.

Calculations in the first type of program above were made for offered loads of direct traffic determined by the formula $B(c,a) = 0.05$. The resulting grade of service of direct traffic and the amount of S/F traffic carried for $r = 83.3$ are presented in Table I. (The grade of service is plotted as a solid line in Figs. 4 and 5.) It can be seen that substantial

TABLE I — $D(c,a, 83.3,s)$ AND b FOR $B(c,a) = 0.05$

c	a	One Trunk in Reserve		Two Trunks in Reserve	
		$D(c,a,83.3,1)$	b	$D(c,a,83.3,2)$	b
2	0.38132	0.05255	0.69127	—	—
3	0.89940	0.05477	1.20463	0.05003	0.41676
4	1.52462	0.05670	1.61853	0.05008	0.79092
5	2.21848	0.05842	1.96954	0.05014	1.11664
6	2.96033	0.05996	2.27714	0.05021	1.40371
7	3.73782	0.06137	2.55294	0.05028	1.66049
8	4.54297	0.06267	2.80439	0.05036	1.89308
9	5.37025	0.06387	3.03662	0.05044	2.10599
10	6.21572	0.06500	3.25327	0.05052	2.30257
12	7.95007	0.06704	3.64998	0.05069	2.65643
16	11.54361	0.07053	4.34111	0.05105	3.25047
20	15.24928	0.07344	4.94398	0.05140	3.74232
24	19.03073	0.07592	5.49001	0.05176	4.16546
30	24.80184	0.07908	6.23847	0.05229	4.71144
36	30.65736	0.08173	6.92986	0.05279	5.18284
48	42.53693	0.08598	8.20641	0.05374	5.98058

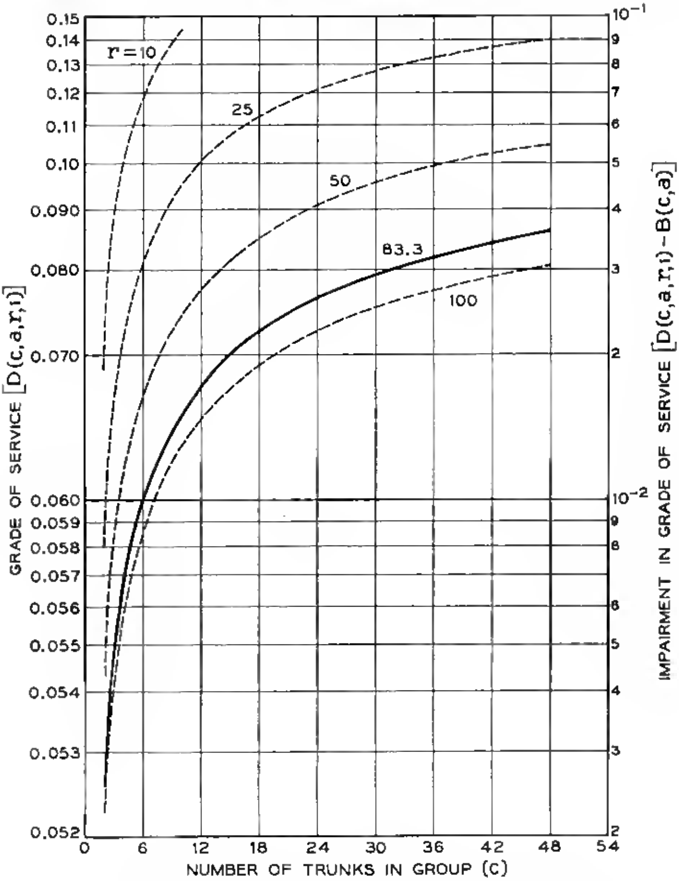

Fig. 4 — Grade of service $D(c,a,r,1)$ with a given by $B(c,a) = 0.05$, vs number of trunks c.

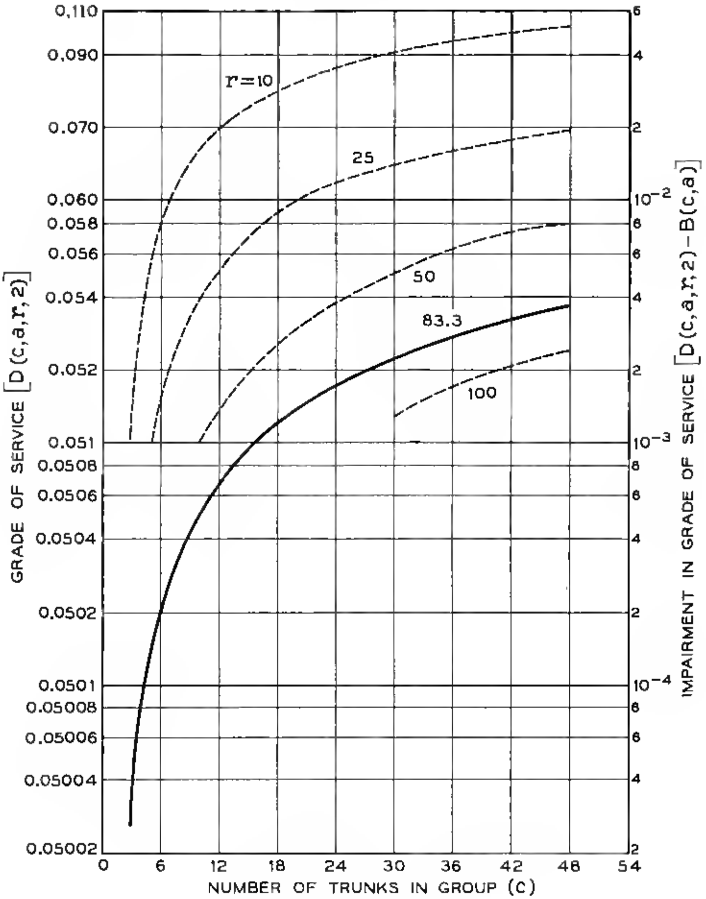

Fig. 5 — Grade of service $D(c,a,r,2)$ with a given by $B(c,a) = 0.05$, vs number of trunks c.

amounts of S/F traffic can be carried. When one trunk is held in reserve, the amount ranges from 0.7 erlang (700 messages per hour) in the case of a trunk group with 2 trunks, to 8.2 erlangs (8200 messages per hour) in the case of a trunk group with 48 trunks. When two trunks are held in reserve, the amount ranges from 0.42 erlang (420 messages per hour) in the case of a trunk group with 3 trunks, to 6.0 erlangs (6000 messages per hour) in the case of a trunk group with 48 trunks.

The impairment in grade of service to direct traffic — i.e., the difference between the grade of service under the dual use of trunks and the

grade of service when no S/F traffic is sent (for the same amount of direct traffic) — is rather small when 1 trunk is held in reserve and practically insignificant when 2 trunks are held in reserve. The impairment increases from 0.003 for the 2-trunk group to 0.036 for the 48-trunk group when 1 trunk is held in reserve, and from 0.00003 for the 3-trunk group to 0.0037 for the 48-trunk group when 2 trunks are held in reserve.

Similar calculations were carried out for different ratios r of holding times, ranging from 100 to 1. The grade of service to the direct traffic is plotted, with r as a parameter, as dashed lines in Figs. 4 and 5. The impairment increases rapidly with decreasing r. If the impairment were plotted as a function of r, it can be seen that for $r > 10$ the impairment obeys closely the asymptotic functional relation $1/r$ of (20) for $s = 1$ and $1/r^2$ of (41) for $s = 2$. For lower r, the impairment increases more slowly than $1/r$ for $s = 1$, or $1/r^2$ for $s = 2$. The capacity b for carrying S/F traffic increases slowly with the increase of probability of loss, which is indicated for $s = 1$ by (19).

Calculations in the second type of program were used to obtain the offered amount of direct traffic, which will result in grade of service 0.05 under the dual use of trunks: i.e., load a in erlangs determined by $D(c,a,83.3,1) = 0.05$ and by $D(c,a,83.3,2) = 0.05$. The amount b of S/F traffic was computed at the same time. These results are presented in Table II and Fig. 6. Comparing dual use of trunks having 1 trunk in reserve with use of trunks for direct traffic only at the same grade of service, it can be seen that for the 2-trunk group, 0.7 erlangs of S/F

TABLE II — a AND b FOR $D(c,a,83.3,s) = 0.05$

c	One Trunk in Reserve $D(c,a,83.3,1) = 0.05$		Two Trunks in Reserve $D(c,a,83.3,2) = 0.05$	
	a	b	a	b
2	0.36953	0.69894	—	—
3	0.85954	1.23344	0.89912	0.41686
4	1.44427	1.67795	1.52364	0.79143
5	2.08803	2.06637	2.21629	1.11790
6	2.77203	2.41657	2.95642	1.40608
7	3.48519	2.73907	3.73167	1.66431
8	4.22041	3.04060	4.53409	1.89868
9	4.97292	3.32572	5.35812	2.11369
10	5.73925	3.59771	6.19983	2.31268
12	7.30392	4.11127	7.92525	2.67221
16	10.52058	5.05545	11.49555	3.28063
20	13.80962	5.93086	15.17145	3.79025
24	17.14272	6.76441	18.91730	4.23396
30	22.19328	7.96639	24.62530	4.81501
36	27.28129	9.13277	30.40758	5.32546
48	37.51678	11.40906	42.11484	6.21013

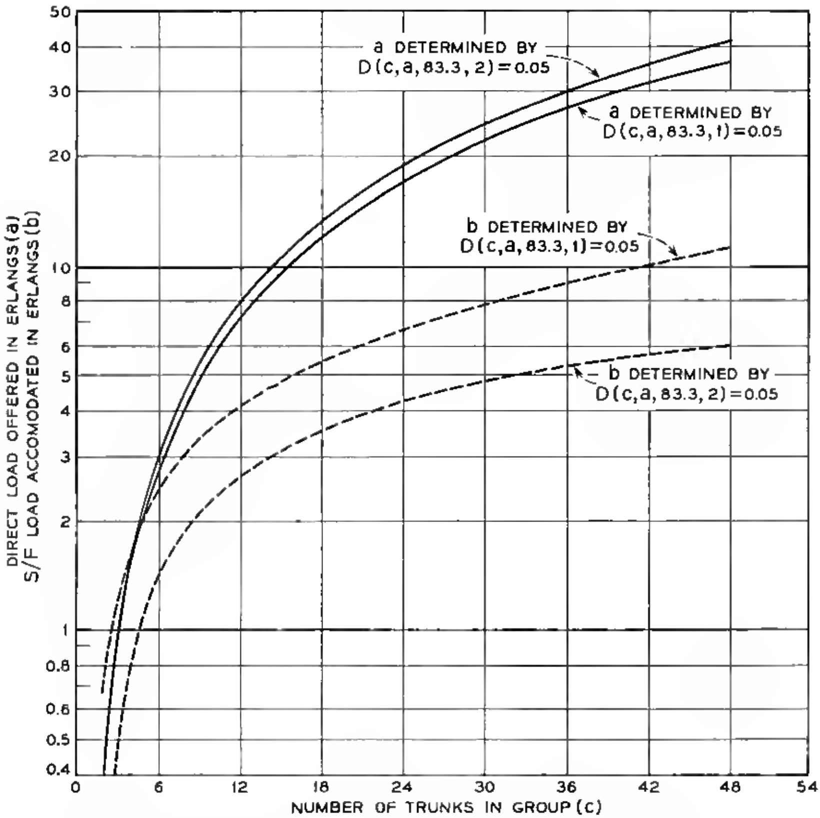

Fig. 6 — Load of S/F traffic b and offered load of direct traffic a defined by $D(c,a,83.3,1) = 0.05$ and by $D(c,a,83.3,2) = 0.05$, vs number of trunks c.

traffic (700 messages per hour) can be sent at the cost of decreasing the offered direct load from 0.381 erlang to 0.369 erlang: i.e. by 0.012 erlang (a fraction 0.14 of the average holding time of a direct call). For the 48-trunk group, 11.4 erlangs of S/F traffic (11,400 messages per hour) can be sent at the cost of decreasing the offered direct load from 42.5 erlangs to 37.5 erlangs: i.e. by 5.0 erlangs (60 direct calls). Making a similar comparison when two trunks are held in reserve, it will be noted that if the offered load a determined by $B(c,a) = 0.05$ were to be plotted in Fig. 6, it would virtually coincide with the plotted offered load a determined by $D(c,a,83.3,2) = 0.05$. Thus, the decrease in direct capacity is negligible. In Table III an example is presented of trunk occupancy probabilities which result from a calculation in the second type of pro-

TABLE III — TRUNK OCCUPANCY PROBABILITIES

z	One Trunk in Reserve $D(5,a,83.3,1) = 0.05$		z	Two Trunks in Reserve $D(5,a,83.3,2) = 0.05$		
	R_z	S_z		R_z	W_z	S_z
0	0.03912	0.09368	0	0.04977	0.11229	0.19610
1	0.00450	0.18807	1	0.00018	0.00688	0.27160
2	0.00338	0.27508	2	0.00004	0.00322	0.24702
3	0.00220	0.26559	3	0.00001	0.00098	0.11190
4	0.00080	0.12758				
Total	0.05000	0.95000		0.05000	0.12337	0.82663

gram; probabilities R_z and S_z for $D(5,a,83.3,1)$ and the probabilities R_z, W_z, and S_z for $D(5,a,83.3,2)$ are given.

These calculations were repeated for different values of r ranging from 100 to 1. The capacity a for carrying direct traffic determined by $D(c,a,r,1) = 0.05$ is plotted in Fig. 7 and by $D(c,a,r,2) = 0.05$ in Fig. 8.

The following conclusions can be drawn from the numerical results. The dual use of long-haul trunks for both direct and S/F traffic can be economically attractive. The calculations indicate that when one or two trunks are held in reserve for possible arrivals of direct traffic, it is possible to send large amounts of S/F traffic on the same trunk groups as direct traffic with little or negligible impairment of the direct traffic, provided the parameters of group size and the number of trunks held in reserve are properly selected. The dual use of trunks is especially attractive if the average holding time of the S/F traffic is short compared with the average holding time of the direct traffic. When the holding time of the S/F traffic approaches the average holding time of the direct traffic, the capacity for carrying direct traffic deteriorates sharply, unless two trunks (or more, for large trunk-groups) are held in reserve.

The potential economics are believed to be of greatest importance in systems with small trunk groups. A manipulation of the data (1 trunk in reserve) will serve to point out that the gain in trunk group efficiency is very substantial in the smaller groups and decreases as the number of trunks increases. The trunk efficiency is the load carried divided by the number of trunks. The load carried is $[1 - B(c,a)]a = 0.95a$ (from Table 1) when no S/F traffic is present, and

$$[1 - D(c,a,r,1)]a + b = 0.95a + b$$

(from Table II) under the dual use of trunks. The 2-trunk group increases in efficiency from 18 to about 52 per cent, the 10-trunk group from 59

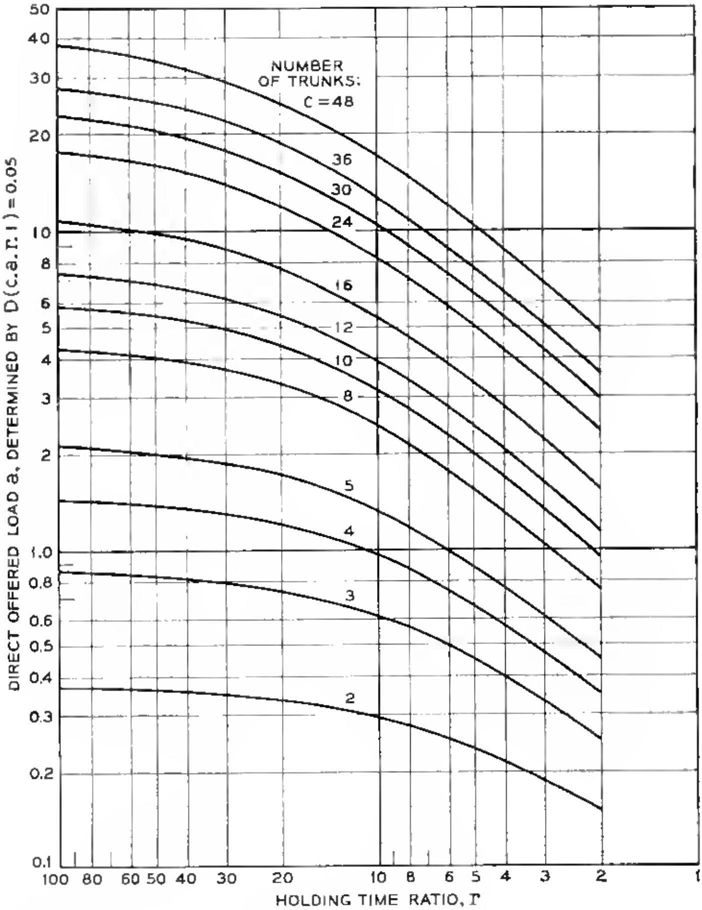

Fig. 7 — Offered load of direct traffic a defined by $D(c,a,r,1) = 0.05$, vs ratio of holding times r.

to about 90 per cent, and the 48-trunk group from 84 to about 98 per cent. These advances are significant in terms of possible savings in long-haul transmission plant.

This method of operating circuit groups at such a high level of occupancy remains to be evaluated in terms of the grade of service of the S/F user. The grade of service of the S/F traffic has not been discussed here.

The analysis given here was confined to a single trunk group carrying

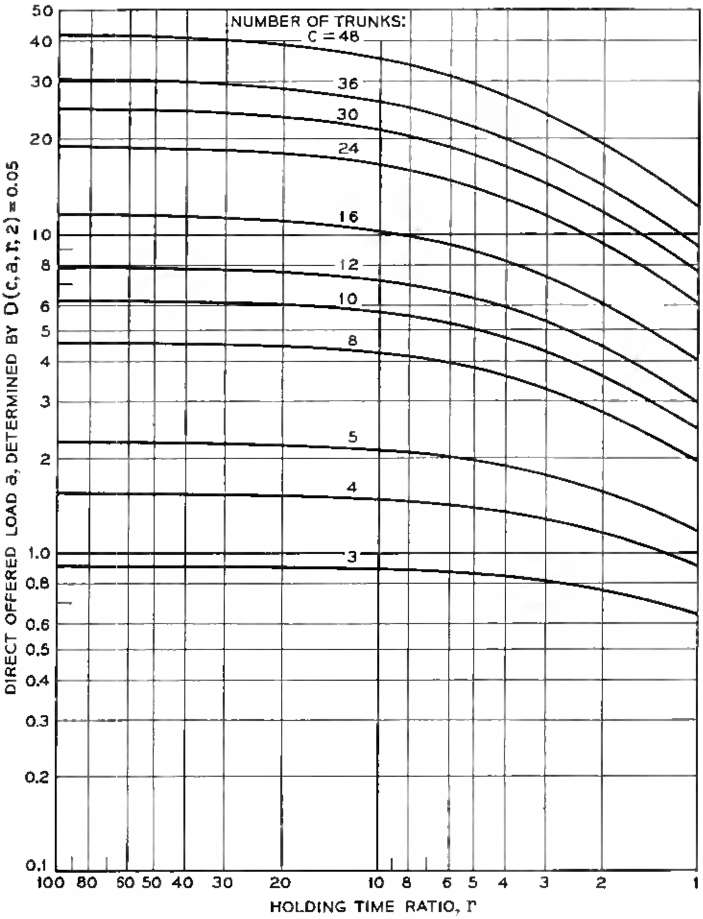

Fig. 8 — Offered load of direct traffic a defined by $D(c,a,r,2) = 0.05$, vs ratio of holding times r.

traffic between two points. The problems of alternate routing and through-switching of traffic are yet to be explored.

IV. ACKNOWLEDGMENT

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APPENDIX

Glossary of Symbols

a	Offered amount of direct traffic, in erlangs, during busy hour.
b	Amount of S/F traffic per busy hour that can be accommodated, in erlangs.
c	Number of trunks in a trunk group.
s	Number of trunks held in reserve.
t_0	Average holding time of S/F traffic.
T	Average holding time of direct traffic.
r	Ratio of the average holding time of direct traffic to the average holding time of S/F traffic, T/t_0.
$B(c,a)$	Grade of service to direct traffic on lost-calls-cleared basis, with c trunks in the trunk-group and an offered load a.
$D(c,a,r,s)$	Grade of service to direct traffic on lost-calls-cleared basis, with c trunks in the trunk-group, offered load a of direct traffic, ratio of holding times r, and s trunks in reserve for possible arrivals of direct traffic.
G_x	Probability of exactly x trunks being occupied by direct traffic.
n	Expected number of arrivals of direct traffic during the busy hour.
R_z	Probability of, or state of, all-trunks-busy, with S/F traffic on z trunks.
S_z	Probability of, or state of, specified number of trunks (one when $s = 1$, two when $s = 2$) in reserve, with S/F traffic on z trunks.
W_z	Probability of, or state of, one trunk in reserve, with S/F traffic on z trunks (under the operating method where $s = 2$).
x	Number of trunks occupied by direct traffic.
z	Number of trunks occupied by S/F traffic.

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